TOTAL DIFFERENTIAL EQUATIONS

Any equation in the form $Pdx + Qdy + Rdz = 0$ is called total diff. equation. where P, Q, R are functions of x, y, z.

The necessary condition of the integrability of total diff. equation.

If $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$

Then given equation is integrable.

Geometrical interpretation: If $\phi(x, y, z)$ is a solution of the equation $Pdx + Qdy + Rdz = 0$ then the direction numbers of the normal to the surface at the point (x, y, z) are P, Q, R.

N.B. First check the integrability condition. If condition is satisfied, then the given equation may be integrable otherwise not. We can solve the equation in two methods.

1. Taking any variable constant
2. By inspection.

Now we shall discuss some problems.

Problem No: 21 Solve: $(yz+2x)dx + (xz+2y)dy + (xy+2z)dz = 0$

Ans: given equation is

$$(yz+2x)dx + (xz+2y)dy + (xy+2z)dz = 0$$

Here $P = yz+2x$, $Q = xz+2y$, $R = xy+2z$

C.O.I. $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (yz+2x)(x-x) + (xz+2y)(y-y) + (xy+2z)(z-z) = 0$$

Hence condition of integrability is satisfied. Therefore given equation is integrable.

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Now given equation is

$$(yz+2x)\,dx + (xz+2y)\,dy + (xy+2z)\,dz = 0$$

~~we solve it by method of inspection.~~

ie $yz\,dx + 2x\,dx + xz\,dy + 2y\,dy + xy\,dz + 2z\,dz = 0$

$\Rightarrow (yz\,dx + xz\,dy + xy\,dz) + 2(x\,dx + y\,dy + z\,dz) = 0$

$\Rightarrow d(xyz) + 2(x\,dx + y\,dy + z\,dz) = 0$

$\Rightarrow \int d(xyz) + 2\left[\int x\,dx + \int y\,dy + \int z\,dz\right] = 0$

$\Rightarrow xyz + 2\left[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right] = 0 + c^2$

$\Rightarrow xyz + x^2 + y^2 + z^2 = c^2$, where c^2 is constant.

Ans

Problem No: 21 Solve: $z(z-y)\,dx + (z+x)z\,dy + x(x+y)\,dz = 0$

Ans Here $P = z(z-y)$, $Q = (z+x)z$, $R = x(x+y)$

C.O.I. is $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= z(z-y)(z+\cancel{x}+z-\cancel{x}) + (z+x)z(x+y+x-z+y-z)$
$\quad + x(x+y)(-z-z)$

$= z(z-y)\cdot 2z + (z+x)z(2x+2y-2z) - 2x(x+y)z$

$= 2z\left[z^2 - yz + (z+x)(x+y-z) - x^2 - xy\right]$

$= 2z\left[\cancel{z^2} - \cancel{yz} + \cancel{zx} + \cancel{yz} - \cancel{z^2} + \cancel{x^2} + \cancel{xy} - \cancel{xz} - \cancel{x^2} - \cancel{xy}\right]$

$= 2z \times 0 = 0$

Hence given equation is integrable.

~~Now we solve it by taking a variable as constant.~~

Let us choose x as constant, then $dx = 0$.

Now the transformed equation is

$$(z+x)z\,dy + x(x+y)\,dz = 0$$

$$\Rightarrow (z+x)z\,dy = -x(x+y)\,dz$$

$$\Rightarrow \frac{dy}{x+y} = -x\frac{dz}{z(z+x)}$$

$$\Rightarrow \frac{dy}{x+y} = +\left[\frac{1}{z+x} - \frac{1}{z}\right]dz$$

$$\Rightarrow \int\frac{dy}{x+y} = \int\frac{1}{z+x}dz - \int\frac{1}{z}dz$$

$$\Rightarrow \log|x+y| = \log|z+x| - \log|z| + \log\phi(x).$$

where $\log\phi(x)$. a constant, regarded as a function of x

$$\Rightarrow \log|x+y| - \log|z+x| + \log|z| = \log|\phi(x)|$$

$$\Rightarrow \log\frac{z(x+y)}{z+x} = \log|\phi(x)|$$

$$\Rightarrow \frac{z(x+y)}{z+x} = \phi(x) \quad (1)$$

, Taking differentials.

$$\Rightarrow \frac{(z+x)d\{z(x+y)\} - z(x+y)\,d(z+x)}{(z+x)^2} = d\phi$$

$$\Rightarrow \frac{(z+x)\{(x+y)dz + z(dx+dy)\} - z(x+y)(dz+dx)}{(z+x)^2} = d\phi$$

$$\Rightarrow \frac{(z+x)(x+y)dz + z(z+x)dx + z(z+x)dy - z(x+y)dz - z(x+y)dx}{(z+x)^2} = d\phi$$

$$\Rightarrow \frac{z(z+\cancel{x}-\cancel{x}-y)dx + z(z+x)dy + (x+y)(\cancel{z}+x-\cancel{z})dz}{(z+x)^2} = d\phi$$

$$\Rightarrow \frac{z(z-y)dx + z(z+x)dy + x(x+y)dz}{(z+x)^2} = d\phi$$

comparing it with given Equation, we have $d\phi = 0$, Integrating, $\phi = c$. Constant.

Putting the value of ϕ in (1), the solution is

$$\frac{z(x+y)}{z+x} = c \Rightarrow z(x+y) = c(z+x)$$

Ans

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