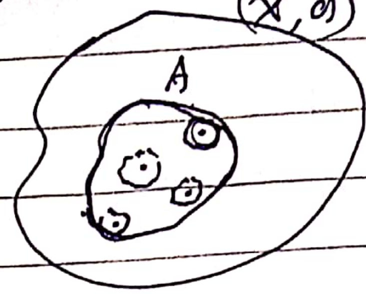
Open Set

A subset A of a metric space (X,d) is said to be open if for every $x \in A$ there exist a real number $r > 0$ such that

$$S_r(x) \subseteq A$$



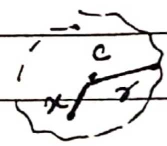
Theorem: $\rightarrow$ Every open sphere in a metric space (X,d) is an open set

Proof Let $c \in X$ and $r > 0$.

$\Rightarrow$ $S_r(c)$ is an open sphere

We want to show that $S_r(c)$ is open set.

Let $x \in S_r(c)$



$\Rightarrow d(x,c) < r$

Let $\lambda = r - d(x,c)$

of course $\lambda > 0$ ✓

Consider $S_\lambda(x)$

We we'll show that $S_\lambda(x) \subseteq S_r(c)$

Let $y \in S_\lambda(x)$

$\Rightarrow d(x,y) < \lambda$

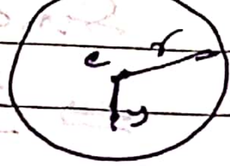
$\Rightarrow d(x,y) < r - d(x,c)$ —— (1)

Now $d(y,c) \leq d(y,x) + d(x,c)$

$$< r - \cancel{d(x,c)} + \cancel{d(x,c)}$$

$\Rightarrow d(y,c) < r$

$\Rightarrow y \in S_r(c)$



$\Rightarrow S_x(x) \subseteq S_r(c)$

Hence $S_r(c)$ is open.

Thm A subset G of a metric space (X,d) is is open if and only if G is union of open sphere

Proof Let G be any open subset of a metric space (X,d)

$\Rightarrow$ For every $x \in G$, $\exists\ r_x > 0$ s.t.

$$S_{r_x}(x) \subseteq G$$

Thus $\bigcup_{x \in G} S_{r_x}(x) = G$

Conversely: Let G be a union open spheres in a metric space (X,d)

ie $G = \bigcup \{S_\alpha\}$ where S_α is open sphere

Let $x \in G$

$\Rightarrow x \in \bigcup S_\alpha$

$\Rightarrow \exists$ Some α_0 s.t

$x \in S_{\alpha_0}$

i.e $x \in S_{\alpha_0} \subseteq G$

Since open sphere is an open set, i.e. S_{α_0} is open set

$\Rightarrow \exists\ r > 0$ s.t

$S_r(x) \subseteq S_{\alpha_0}$ & $S_{\alpha_0} \subseteq G$

On Combining $S_r(x) \subseteq G$

$\Rightarrow G$ is open.

Theorem:

Let (X,d) be any metric space then

1. X and ϕ are always open
2. Arbitrary union of open sets is open
3. Finite intersection of open sets is open.

Proof (1) By def^n of open set, ϕ is always open

X is open

$\because$ Every $x \in X$, $S_r(x) \subset X$

(2) Let $\{G_\lambda : \lambda \in \Delta\}$ be family of open sets in a metric space (X,d)

Let $G = \bigcup_{\lambda \in \Delta} G_\lambda$

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We want to show that G is open.
If G is empty set then G is open. So let G is non-empty
Let $x \in G$

$\Rightarrow x \in \bigcup_{\lambda \in \Delta} G_\lambda$

$\Rightarrow x \in G_\lambda$ for some $\lambda \in \Delta$

Since G_λ is open

$\Rightarrow \exists\ r > 0$ s.t. $S_r(x) \subseteq G_\lambda$

$\Rightarrow S_r(x) \subseteq \bigcup_{\lambda \in \Delta} G_\lambda$

$\Rightarrow S_r(x) \subseteq G$

Hence G is open.

(3) Let $\{G_1, G_2, \dots G_m\}$ be a finite family of open sets in a metric space (X, d)

Let $G = \bigcap_{\lambda=1}^{m} G_\lambda$

We want to show that G is open.

Let $x \in G$

$\Rightarrow x \in \bigcap_{\lambda=1}^{m} G_\lambda$

$\Rightarrow x \in G_\lambda$ for $\lambda = 1, 2, 3, \dots m$

$\Rightarrow \exists\ r_\lambda > 0$ s.t

$S_{r_\lambda}(x) \subseteq G_\lambda \ \forall\ \lambda = 1, 2, 3, \dots m$

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$$r = \min\{r_1, r_2, \cdots r_m\}$$

So

$$S_r(x) \subseteq S_{r_1}(x) \subseteq G_1 \quad (\because r < r_1)$$

$$S_r(x) \subseteq S_{r_2}(x) \subseteq G_2$$

$$\cdots\cdots\cdots$$

$$S_r(x) \subseteq S_{r_m}(x) \subseteq G_m$$

Taking intersection (omitting middle sphere)

$$S_r(x) \subseteq \bigcap_{\lambda=1}^{m} G_\lambda$$

$$\Rightarrow S_r(x) \subseteq G$$

Hence G is open. Proved