Date			
Page No.			

More Examples on Metric Space

① If (X, d) is a metric space, then prove that (X, ρ) is also a metric space where

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

Proof

Since $d(x,y) \geq 0 \quad \forall\, x,y \in X$

$\Rightarrow 1+d(x,y) > 0$

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} \geq 0$$

$\Rightarrow \rho(x,y) \geq 0 \quad \forall\, x,y \in X$ —— ①

Also $d(x,y) = 0$ if & only if $x = y$

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} = 0 \text{ if \& only if } x = y$$

$\Rightarrow \rho(x,y) = 0$ if & only if $x = y$ —— ②

Now
$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)} \quad (\because d(x,y) = d(y,x))$$

$= \rho(y,x)$ —— ③

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Since d is metric on X

$$\Rightarrow d(x,z) \leq d(x,y) + d(y,z) \quad \text{--- (i)}$$

Now $$\frac{1}{d(x,z)} \geq \frac{1}{d(x,y)+d(y,z)}$$

$$\Rightarrow 1+\frac{1}{d(x,z)} \geq 1+\frac{1}{d(x,y)+d(y,z)}$$

$$\Rightarrow \frac{1}{1+\frac{1}{d(x,z)}} \leq \frac{1}{1+\frac{1}{d(x,y)+d(y,z)}}$$

$$\Rightarrow \frac{d(x,z)}{1+d(x,z)} \leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\Rightarrow \rho(x,z) \leq \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\Rightarrow \rho(x,z) \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$\Rightarrow \rho(x,z) \leq \rho(x,y) + \rho(y,z) \quad \text{--- (4)}$$

Hence (X,ρ) is also a metric space

Date
Page No.

(2) Let (X, d) be a metric space, then prove that (X, ρ) is also a metric space where

$$\rho(x,y) = \min\{1, d(x,y)\}$$

Sol^n Here $\rho(x,y) = \min\{1, d(x,y)\}$

i.e. $$\rho(x,y) = \begin{cases} 1 & \text{if } d(x,y) \geq 1 \\ d(x,y) & \text{if } d(x,y) < 1 \end{cases}$$

By def^n of ρ, it is clear that

$\rho(x,y) \geq 0 \quad \forall\, x, y \in X$ —— (1)

$\rho(x,y) = 0$ iff $d(x,y) < 1$
iff $d(x,y) = 0$ $\{\because \rho(x,y) = d(x,y)\}$
iff $x = y$ —— (2)

(i) Also if $\rho(x,y) = 1$ i.e. $d(x,y) \geq 1$
$\Rightarrow d(y,x) \geq 1$
$\Rightarrow \rho(y,x) = 1$ i.e. $\rho(x,y) = \rho(y,x)$

(ii) if $\rho(x,y) < 1$ i.e. $d(x,y) < 1$
$\rho(x,y) = d(x,y)$
$= d(y,x)$
$= \rho(y,x)$ $(\because d(y,x) < 1)$

$\Rightarrow$ Symmetry holds

Now we want to show that

$$\rho(x,y) + \rho(y,z) \geq \rho(x,z)$$

Case I If $\rho(x,y) = 1$ $\rho(y,z) = 1$

Obviously $\rho(x,z) \leq \rho(x,y) + \rho(y,z) = 2$

$(\because \rho(x,z) \leq 1)$

Case II If $\rho(x,y) < 1$ & $\rho(y,z) < 1$

$\Rightarrow \rho(x,y) = d(x,y)$ & $\rho(y,z) = d(y,z)$

$$\rho(x,z) = \min\{1, d(x,z)\}$$

$$\leq d(x,z)$$

$$\leq d(x,y) + d(y,z)$$

$$= \rho(x,y) + \rho(y,z)$$

ie $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

Hence (X, ρ) is also metric space

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Bounded Subset

A subset A of a metric space (X,d) is said to be bounded if $\exists\ a \in A$ and $K > 0$ such that

$$d(x,a) \leq K \quad \forall\ x \in A$$

Bounded Metric space

A metric space (X,d) is said to be bounded if $\exists\ K > 0$ such that

$$d(x,y) \leq K \quad \forall\ x, y \in X$$

Thm Finite union of bounded sets is bounded

Proof Let $A_1, A_2, \dots A_m$ be bounded subsets of a metric space (X,d)

$$\Rightarrow \exists\ a_i \in A_i \ \& \ k_i > 0 \quad (i = 1, 2, 3, \dots m)$$

Such that $d(x, a_i) \leq k_i \ \forall\ i = 1, 2, \dots m$

Let $A = \bigcup_{i=1}^{m} A_i$ and let $x \in A$

$\Rightarrow x \in A_i$ for at least one $i \in I$

Clearly $a_1 \in A$

Now $d(x, a_1) \leq d(x, a_i) + d(a_i, a_1) \ \forall i$

$\leq k_i + d(a_i, a_1) \ \forall i$

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$$\leq \max_{i=1} K_i + \max_i (d_i\ a_1)$$

$$= K$$

$\Rightarrow A$ is bounded (proved)

Def^n **Diameter:**

Let A is subset of a metric space (X,d) then diameter of A is given

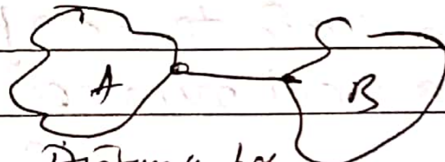
$$\delta(A) = l.u.b\{d(x,y) \mid x,y \in A\}$$

is distance between farthest points

Def^n Distance between two non-empty sets

Let A, B be any two non-empty subsets of a metric space (X,d). Then distance between A and B is given by

$$d(A,B) = g.l.b\{d(x,y) \mid x \in A, y \in B\}$$



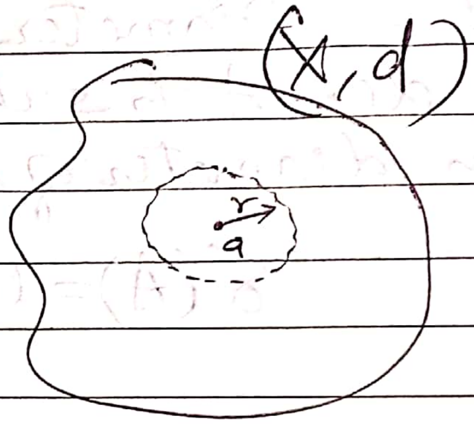
Distance between nearest points from A & B.

Date | Page No.

Defn. Open Sphere

Let (X,d) be any metric space and $a \in X$, $r>0$ then an open sphere centred at a, with radius r is given by

$$S_r(a) = \{x \in X \mid d(x,a) < r\}$$



Obviously $S_r(a) \subset X$

$S_r(a)$ is also denoted by $S(a,r)$

It is also called "open ball"

Defn Open Set

A subset A of a metric space (X,d) is said to be open if for every $x \in A$, $\exists$ a real number $r>0$ such that

$$S_r(x) \subseteq A$$