Metric Space

A pair (X, d) is said to be a metric space if X is a ~~set~~ non empty set and d is a function from $X \times X$ into R such that

M_1: $d(x,y) \geq 0 \quad \forall\, x, y \in X$

M_2: $d(x,y) = 0$ if and only if $x = y$

M_3: $d(x,y) = d(y,x)$

M_4: $d(x,z) \leq d(x,y) + d(y,z)$

$M_3 \rightarrow$ Symmetry & M_4 is triangular inequality

Pseudo Metric space

A pair (X, d) is said to be a pseudo-metric space if X is a non-empty set and $d: X \times X \to R$ be any function such that

M_1: $d(x,y) \geq 0 \quad \forall\, x, y \in X$

M_2: $x = y \Rightarrow d(x,y) = 0$

M_3: $d(x,y) = d(y,x)$

M_4: $d(x,z) \leq d(x,y) + d(y,z)$

d is called "Metric" or distance function.

Examples

① ~~Discrete metric space~~.

Let X be any non-empty set and $d: X \times X \to R$ be any function defined as

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that (X,d) is a metric space

Sol^n

From definition

$$d(x,y) \geq 0 \quad \forall\, x,y \in X \quad \text{—} ①$$

Now let $d(x,y) = 0$

$$\Leftrightarrow \quad x = y \quad (\text{From def}^n) \text{ — } ②$$

Also $d(x,y) = d(y,x)$ ——— ③

$\because$ If $d(x,y)=0 \Rightarrow x=y$

$\Rightarrow y=x$

$\Rightarrow d(y,x)=0$ ie $d(x,y)=d(y,x)$

If $d(x,y)=1 \Rightarrow x \neq y$

$\Rightarrow y \neq x$

$\Rightarrow d(y,x)=1$ ie $d(x,y)=d(y,x)$

Symmetry holds.

Case I

∵ Let $d(x,z)=1 \Rightarrow x \neq z$

Also $d(x,y)+d(y,z) = 1$ or 2

∵ either $y \neq x$ or $y \neq z$ or both

$\Rightarrow d(x,z) \leq d(x,y)+d(y,z)$

Case II Let $d(x,z)=0 \Rightarrow x=z$

Also $d(x,y)+d(y,z) = 0$ or 2

∵ Either $y=x=z$ or $y \neq x$ & $y \neq z$

$\Rightarrow d(x,z) \leq d(x,y)+d(y,z)$

So triangular inequality holds

$\Rightarrow (X,d)$ is a metric space

(2) Usual Metric space

Let R be set of real numbers and $d: R \times R \rightarrow R$ is a function defined as

$$d(x,y)=|x-y|$$

∴ Show that (R,d) is a metric space

Sol^n

Since $|x-y| \geq 0 \quad \forall x,y \in R$

$\Rightarrow d(x,y) \geq 0 \quad \forall x,y \in R$ — (1)

Teacher's Signature..........

Date
Page No.

Now $d(x,y) = 0$

$\Leftrightarrow |x-y| = 0$

$\Leftrightarrow x-y = 0$

$\Leftrightarrow x = y$ ——— (2)

Also $d(x,y) = |x-y|$

$= |y-x|$

$= d(y,x)$ ——— (3)

And $d(x,z) = |x-z|$

$= |x-y+y-z|$

$\leq |x-y| + |y-z|$

$= d(x,y) + d(y,z)$

i.e. $d(x,z) \leq d(x,y) + d(y,z)$ ——— (4)

Hence (R,d) is a metric space

(3) Let R^n be set of all n-tuples where $x \in R^n$ mean $x = (x_1, x_2, \cdots x_n)$

We define a function as

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Show that (R^n, d) is a metric space

Teacher's Signature.....................

Date
Page No.

Sol^n Since $(x_i - y_i)^2 \geq 0 \quad \forall x_i, y_i$

$$\Rightarrow \sum_{i=1}^{n} (x_i - y_i)^2 \geq 0 \quad \forall x, y \in R^n$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)} \geq 0 \quad \forall x, y \in R^n$$

$$\Rightarrow \quad d(x,y) \geq 0 \quad \forall x, y \in R^n \quad \text{—— (1)}$$

Let $d(x,y) = 0$

$$\Leftrightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} (x_i - y_i)^2 = 0$$

$$\Leftrightarrow \quad (x_i - y_i) = 0 \quad \forall i = 1, 2 \ldots n$$

$$\Leftrightarrow \quad x_i = y_i \quad \forall i = 1, 2, \ldots n$$

$$\Leftrightarrow \quad x = y \quad \text{—— (2)}$$

Now $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$$= \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

$$= d(y,x) \quad \text{—— (3)}$$

Teacher's Signature..........................

From Cauchy Schwarz inequality

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

$$\Rightarrow \sum_{i=1}^{n} u_i v_i \leq \sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2} \quad \text{——} (1)$$

Now $\sum_{i=1}^{n}(u_i+v_i)^2 = \sum_{i=1}^{n}(u_i^2+v_i^2+2u_iv_i)$

$$= \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sum_{i=1}^{n} u_i v_i$$

$$\leq \left(\sqrt{\sum_{i=1}^{n} u_i^2}\right)^2 + \left(\sqrt{\sum_{i=1}^{n} v_i^2}\right)^2 + 2\sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2}$$

[Using (1)]

$$= \left(\sqrt{\sum_{i=1}^{n} u_i^2} + \sqrt{\sum_{i=1}^{n} v_i^2}\right)^2$$

Taking square root

$$\sqrt{\sum_{i=1}^{n}(u_i+v_i)^2} \leq \sqrt{\sum_{i=1}^{n} u_i^2} + \sqrt{\sum_{i=1}^{n} v_i^2}$$

Putting $u_i = x_i - y_i$ & $v_i = y_i - z_i$

$$\Rightarrow \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}$$

ie $d(x,z) \leq d(x,y) + d(y,z)$ —— (4)

$\Rightarrow (\mathbb{R}^n, d)$ is a metric space

Teacher's Signature............