

## Order relation in Boolean algebras

Let  $a$  and  $b$  be any two elements in a Boolean algebra  $B$ , then if  $ab' = 0$  then we say that  $a$  is inserted in  $b$  and we denote this by  $a \leq b$

i.e.  $a \leq b$  if & only if  $ab' = 0 \quad \forall a, b \in B$ .

Th<sup>m</sup> In a Boolean algebra  $B$ , the order relation " $\leq$ " is a partial order relation.

Proof

Let  $a \in B$

$$\Rightarrow a \cdot a' = 0$$

$$\Rightarrow a \leq a$$

$\Rightarrow$  " $\leq$ " is reflexive relation.

Let  $a, b \in B$  and  $a \leq b$  &  $b \leq a$

$$\Rightarrow ab' = 0 \text{ & } ba' = 0.$$

Now  $b = b + 0$

$$= b + ab'$$

$$= (b + a) \cdot (b + b')$$

$$= (b + a) \cdot 1$$

$$= b + a \quad \text{--- (1)}$$

$$\text{& } a = a + 0$$

$$= a + ba'$$

$$= (a + b) \cdot (a + a')$$

$$= (a + b) \cdot 1$$

$$= a + b$$

$$= b + a \quad \text{--- (2)}$$

From ① & ②  $a=b$

Hence " $\leq$ " is antisymmetric relation.

Let  $a \leq b$  &  $b \leq c$

$$\Rightarrow ab' = 0 \text{ \& } bc' = 0 \quad \text{--- ①}$$

Now

$$a \cdot c' = (a \cdot c) \cdot 1$$

Identity.

$$= (a \cdot c) \cdot (b + b')$$

Complement

$$= (a \cdot c) \cdot b + (a \cdot c) \cdot b'$$

Distributive

$$= (a \cdot c) \cdot b + (c' \cdot a) \cdot b'$$

Commutative

$$= a \cdot (c' \cdot b) + c' \cdot (a \cdot b')$$

Associative

$$= a \cdot (b \cdot c') + c' \cdot (a \cdot b')$$

Commutative

$$= a \cdot 0 + c' \cdot 0$$

From ①

$$= 0 + 0$$

Boundedness

$$= 0$$

$$\Rightarrow a \leq c$$

Hence " $\leq$ " is transitive relation

Hence  $\leq$  is a partial order relation

(Prove).

### Boolean algebra as lattice

As every Boolean algebra satisfies the commutative, associative and absorption laws. This shows that B is also a lattice where  $+$  &  $\cdot$  are join ( $\vee$ ) and meet ( $\wedge$ ) operations respectively.



Ex<sup>m</sup>

Let  $B$  is a Boolean Algebra, then  $(B, \leq)$  when  $\leq$  is defined as  $a \leq b$  if and only if  $ab' = 0$ , is a lattice. Also  $0$  and  $1$  are least and greatest elements of this lattice.

Proof

We have already proved that  $(B, \leq)$  is a Partial Order Set (POSET). We will show that join & meet of any two elements exists.

i.e. we will show that  $a \vee b$  &  $a \wedge b$  exists for all  $a, b \in B$ .

$$\text{Consider } a \cdot (a+b)' = a(a'b')$$

$$= (a \cdot a') \cdot b'$$

$$= 0 \cdot b'$$

$$= 0$$

$$\Rightarrow a \leq (a+b) \text{ --- (1)}$$

Also

$$b \cdot (a+b)' = b(a'b')$$

$$= b \cdot (b'a')$$

$$= (b \cdot b') \cdot a'$$

$$= 0 \cdot a'$$

$$= 0$$

$$\Rightarrow b \leq (a+b) \text{ --- (2)}$$

Hence  $a+b$  is upper bound of  $a$  &  $b$

Let  $c$  is any upper bound of  $a$  &  $b$

$$\Rightarrow a \leq c \text{ \& } b \leq c$$

$$\Rightarrow a \cdot c' = 0 \text{ \& } b \cdot c' = 0$$

$$\Rightarrow a \cdot c' + b \cdot c' = 0$$

$$\Rightarrow (a+b) \cdot c' = 0$$

$$\Rightarrow a+b \leq c$$

Hence  $a+b$  is l.u.b of  $a$  &  $b$

$$\text{i.e. } a \vee b = a+b$$

Similarly we can show that  $a \wedge b = a \cdot b$   
(Use duality)

$$\text{Let } a \in B \Rightarrow a' \in B$$

$$\text{Since } 0 \cdot a' = 0$$

$$\Rightarrow 0 \leq a$$

Since  $a$  is arbitrary,  $0$  is least element in  $B$

$$\text{Similarly let } a \in B$$

$$\text{We have } a \cdot 1' = a \cdot 0$$

$$= 0$$

$$\Rightarrow a \leq 1$$

So  $1$  is greatest element in  $B$ .

Hence  $B$  is a bounded lattice

We know that Boolean algebra satisfies distributive property & complementation property  
Hence

$B$  as a lattice is Bounded Complemented and distributive lattice.